

Black Holes and Biophysical (Mem)-branes

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We argue that the effective theory describing the long-wavelength dynamics of black branes is the same effective theory that describes the dynamics of biophysical membranes. We improve the phase structure of higher-dimensional black rings by considering finite thickness corrections in this effective theory, showing a striking agreement between our analytical results and recent numerical constructions while simultaneously drawing a parallel between gravity and the effective theory of biophysical membranes.

Introduction. In the past three years many remarkable properties of higher-dimensional black holes were uncovered, leading to a new perspective on black holes as materials describable by effective theories of continuous media. In particular, in a long-wavelength regime, besides having fluid-like properties, they can behave as elastic (mem)-branes if bent and exhibit piezoelectric behaviour if charged and flexed. In this regime they are characterised by a set of transport coefficients such as shear and bulk viscosities [1, 2], the Young modulus [3] and the piezoelectric moduli [4], which can be directly measured from gravity.

Fluid and (mem)-brane elastic behaviour of a physical system can be described by a single unified framework of hydrodynamics on embedded surfaces [5, 6]. When applied to black holes and black branes, this framework is commonly known as the blackfold approach [7, 8], which has enabled a systematic scan of new horizon topologies in higher-dimensions [9] and the construction of new approximate solutions, such as black rings in arbitrary space-time dimensions [10].

The dynamics of elastic membranes has also taken an important role in the context of biophysics, beginning with the early work of Helfrich [11] and Canham [12] in the 60's. Considering bending deformations of elastic membranes played a crucial role in the understanding of the shape of red blood cells, explicable in terms of the Helfrich-Canham bending energy, consisting of adding a piece of the form

$$\mathcal{F}_{\text{HC}}[X^\mu(\sigma^a)] = \alpha \int_A dA K^2 \quad , \quad (1)$$

to the free energy of a biophysical membrane. Here $X^\mu(\sigma^a)$ describes the position of the membrane in its involving space, σ^a are a set of coordinates on the membrane, A its area and K its mean extrinsic curvature. The addition of such terms (1) has lead to an exhaustive study of this fluid-elastic system [13].

Black holes in the long-wavelength regime are described by an effective fluid living on a (mem)-brane [8]. Focusing on stationary configurations of this effective fluid, and hence on stationary black hole solutions, it is

possible, using equilibrium partition function techniques for hydrodynamics [14, 15], to construct an effective theory for fluids living on (mem)-branes in a derivative expansion. The resulting effective free energy to second order in the expansion, assuming no transverse spin, can be written as [5]

$$\begin{aligned} \mathcal{F}[X^\mu(\sigma^a)] = & - \int_{\mathcal{B}_p} dV_{(p)} R_0 \left(P + v_1 \omega^{ab} \omega_{ab} + v_2 \mathcal{R} \right. \\ & + v_3 u^a u^b \mathcal{R}_{ab} + \lambda_1 K^i K_i + \lambda_2 K^{abi} K_{abi} \\ & \left. + \lambda_3 u^a u^b K_a^{ci} K_{bci} \right) \quad , \end{aligned} \quad (2)$$

generalizing the Helfrich-Canham bending energy (1) to Lorentzian membranes of arbitrary codimension and accounting for the fact that the fluid living on the membrane can be in a stationary motion. In Eq. (2) we have introduced the fluid pressure P , the fluid velocity u^a and a set of transport coefficients $v_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3$ which only depend on the local fluid temperature and chemical potentials. Furthermore, \mathcal{B}_p denotes the p -dimensional submanifold of the $(p+1)$ -dimensional world volume \mathcal{W}_{p+1} of the brane, $dV_{(p)}$ its infinitesimal volume and R_0 the redshift factor defined via $d^{p+1}\sigma \sqrt{-\gamma} = R_0 d\sigma^0 dV_{(p)}$ with γ being the determinant of the induced metric $\gamma_{ab} = u_a^\mu u_b^\nu g_{\mu\nu}$, where $u_a^\mu = \partial_a X^\mu$ and $g_{\mu\nu}$ the D -dimensional ambient metric.

Besides the bending energy (1), the effective theory (2) accounts for corrections describing the response of the fluid due to a non-zero vorticity ω_{ab} , the possibility of non-zero world volume Ricci scalar \mathcal{R} and tensor \mathcal{R}_{ab} as well as possible elastic (mem)-brane effects due to a non-vanishing extrinsic curvature tensor $K_{ab}^i = n^i_\mu \nabla_a u_b^\mu$, where n^i_μ is an orthogonal projector to the brane, and a non-vanishing mean extrinsic curvature $K^i = \gamma^{ab} K_{ab}^i$. However, despite the fact that the effective theory (2) is characterised by six transport coefficients, for the case of black rings, as we shall see, only one is necessary and the only correction in (2) is the generalisation to arbitrary codimension of the bending energy (1).

These effects are present in large classes of higher-dimensional black holes and our aim in this brief note

is to exemplify how the effective theory (2) can be used to obtain generic features about large classes of black holes. In particular we will see that, similarly to the fact that (1) allowed for biophysical membrane configurations with increased surface area, the elastic effects of (2) result in an increased area (entropy) of black rings for a given value of their angular momenta. These results are in striking agreement with those obtained recently by solving Einstein's equations numerically, therefore confirming the veracity of the effective theory (2) for higher-dimensional black holes and hence drawing a novel parallel between effective theories of black holes and the effective theories of biophysical membranes.

Mem-(brane) dynamics and thermodynamics.

The dynamics of the (mem)-branes described by the effective theory (2) can be obtained by varying (2) with respect to the induced metric and extrinsic curvature tensor. The resulting equations of motion are [5]

$$\nabla_a T^{ab} = u^b_\mu \nabla_a \nabla_b \mathcal{D}^{ab\mu} + \mathcal{D}^{aci} R^b_{aic} , \quad (3)$$

$$T^{ab} K_{ab}{}^i = n^i_\mu \nabla_a \nabla_b \mathcal{D}^{abi} + \mathcal{D}^{abj} R^i_{ajb} , \quad (4)$$

where $R_{\mu\nu\lambda\rho}$ is the background Riemann tensor and where we have defined

$$T^{ab} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta \mathcal{F}}{\delta \gamma_{ab}} , \quad \mathcal{D}^{ab}{}_i = -\frac{1}{\sqrt{-\gamma}} \frac{\delta \mathcal{F}}{\delta K_{ab}{}^i} . \quad (5)$$

Throughout this note, we use the indices a, b, \dots to label the $p+1$ directions along the brane and i, j, k, \dots to label the $n+2$ directions transverse to the brane. The greek indices μ, ν, \dots label generic space-time indices along the $D = n + p + 3$ coordinates. In Eq. (5) we have introduced the stress-tensor of the brane T^{ab} and its bending moment \mathcal{D}^{abi} with the form

$$\mathcal{D}^{abi} = \mathcal{Y}^{abcd} K_{cd}{}^i , \quad (6)$$

where the tensor structure \mathcal{Y}^{abcd} is the Young modulus of the (mem)-brane and reads

$$\mathcal{Y}^{abcd} = 2 \left(\lambda_1 \gamma^{ab} \gamma^{cd} + \lambda_2 \gamma^{a(c} \gamma^{d)b} + \lambda_3 u^{(a} \gamma^{b)(c} u^{d)} \right) , \quad (7)$$

encoding the Hookean response of the (mem)-brane to bending deformations.

Eq. (3) is a non-trivial identity which encodes stress-energy conservation while Eq. (4) encodes the non-trivial elastodynamics of the fluid. Eqs. (3)-(4) are relativistic generalisations of the equations of motion for thin elastic membranes of arbitrary codimension [5]. In the case where bending deformations are ignored, i.e. $\mathcal{D}^{abi} = 0$, they have been shown to arise as constraint equations directly from Einstein equations by bending black branes [10, 16].

The free energy (2) and the equations of motion (3)-(4) are invariant under field redefinitions that displace

the world volume of the brane \mathcal{W}_{p+1} by a small amount $\tilde{\varepsilon}^i$, i.e.,

$$X^i(\sigma^a) \rightarrow X^i(\sigma^a) + \tilde{\varepsilon}^i , \quad (8)$$

inducing the transformations [6]

$$\delta T^{ab} = T^{ab}_{(0)} \tilde{\varepsilon}^i K_i - E^{abcd} K_{cd}{}^i \tilde{\varepsilon}_i , \quad \delta \mathcal{D}^{abi} = T^{ab}_{(0)} \tilde{\varepsilon}^i , \quad (9)$$

where we have split the stress-energy tensor (5) into a perfect fluid (leading order) part $T^{ab}_{(0)}$ and a second order correction $\Pi^{ab}_{(2)}$ proportional to the transport coefficients introduced in (2) according to $T^{ab} = T^{ab}_{(0)} + \Pi^{ab}_{(2)}$. E^{abcd} is the leading order elasticity tensor, defined as [17]

$$T^{ab}_{(0)} = P \gamma^{ab} - P' \mathbf{k} u^a u^b , \quad E^{abcd} = -2 \frac{\partial T^{ab}_{(0)}}{\partial \gamma_{cd}} , \quad (10)$$

where we have used the fact that for stationary fluid configurations the fluid velocity must be aligned with a world volume Killing vector field \mathbf{k}^a such that $u^a = \mathbf{k}^a / \mathbf{k}$ with $\mathbf{k} = |-\gamma_{ab} \mathbf{k}^a \mathbf{k}^b|$ and that the local temperature depends on the global temperature T via $\mathcal{T} = T / \mathbf{k}$. The prime in Eq. (10) denotes the derivative with respect to \mathbf{k} .

Generically, for any stationary fluid configuration we can write the Killing vector field \mathbf{k}^a as

$$\mathbf{k}^a = \xi^a + \Omega^{(b)} \chi^a_{(b)} , \quad (11)$$

where ξ^a denotes a timelike Killing vector field associated with space-time translations, $\chi^a_{(b)}$ denotes a set of space-like Killing vectors fields associated with rotational symmetries and $\Omega^{(b)}$ the corresponding angular velocities. In terms of these Killing vectors we can write all conserved quantities associated with the fluid configuration to all orders in a derivative expansion. In order to do so, we note that, focusing on uncharged configurations, we can write the free energy (2) as

$$\mathcal{F} = M - TS - \Omega^{(a)} J_{(a)} , \quad (12)$$

where M represents the energy of the fluid, S the entropy and $J^{(a)}$ the angular momentum associated with each rotational isometry. Requiring the variation of the free energy (2) to vanish leads to the first law of thermodynamics

$$dM = T dS + \Omega^{(a)} dJ_{(a)} , \quad (13)$$

and hence, defining $\mathcal{F} = - \int_{\mathcal{B}_p} dV_{(p)} R_0 \mathcal{L}$, we obtain the thermodynamic expressions

$$M = - \int_{\mathcal{B}_p} dV_{(p)} R_0 \left(\mathcal{L} + \xi^a \frac{\partial \mathcal{L}}{\partial \mathbf{k}^a} \right) , \quad (14)$$

$$J_{(a)} = - \left(\frac{\partial \mathcal{F}}{\partial \Omega_{(a)}} \right)_T = \int_{\mathcal{B}_p} dV_{(p)} R_0 \chi^b_{(a)} \frac{\partial \mathcal{L}}{\partial \mathbf{k}^b} , \quad (15)$$

$$S = - \left(\frac{\partial \mathcal{F}}{\partial T} \right)_{\Omega_{(a)}} = - \frac{1}{T} \int_{\mathcal{B}_p} dV_{(p)} R_0 \mathbf{k}^a \frac{\partial \mathcal{L}}{\partial \mathbf{k}^a} . \quad (16)$$

The proof of these formulas and the equivalence of (13) with the equations of motion (3)-(4) will be presented in a future publication [18].

Biophysical membrane effect in black rings. We now wish to apply the effective theory (2) to the case of higher-dimensional black rings in asymptotically flat space in the thin ring limit. This corresponds to finite thickness corrections in the blackfold approach going beyond the infinitely thin ring approximation of [10]. In such case the leading order pressure P reads [8]

$$P = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n, \quad r_0 = \frac{n}{4\pi T} \mathbf{k}, \quad (17)$$

where r_0 is the horizon thickness of the black brane and $\Omega_{(n+1)}$ the volume of an $(n+1)$ -sphere. We place the ring in flat space-time parametrised as

$$ds^2 = -dt^2 + dr^2 + r^2 d\psi^2 + \sum_{i=1}^{D-3} dx_i^2, \quad (18)$$

by choosing the embedding functions $X^t = \tau$, $X^r = R$, $X^\psi = \phi$, $X^i = 0$. With this choice the world volume is manifestly flat and hence terms proportional to world volume curvatures in (2) vanish. Furthermore the Killing vector field is chosen such that $\mathbf{k}^a = \partial_\tau + \Omega \partial_\phi$ for constant Ω , leading to a vanishing vorticity ω_{ab} . Using Gauss-Codazzi equations, the leading order equation of motion and the field redefinition (8), black rings are only described by one single transport coefficient to second order [5, 6],

$$\tilde{\lambda}_1 = \frac{\Omega_{(n+1)}}{16\pi G} r_0^{n+2} \frac{(n+1)(3n+4)}{2n^2(n+2)} \xi(n), \quad (19)$$

where $\tilde{\lambda}_1 = \lambda_1 + \lambda_2 + (1/n)\lambda_3$ and

$$\xi(n) = \frac{n \tan(\pi/n)}{\pi} \frac{\Gamma(\frac{n+1}{n})^4}{\Gamma(\frac{n+2}{n})^2}, \quad n \geq 3. \quad (20)$$

Since that the only non-vanishing extrinsic curvature component is $K_{\phi\phi}^r = -R$, the free energy (2) for black rings is thus

$$\mathcal{F}[R] = -2\pi R \left(P + \tilde{\lambda}_1 K^i K_i \right), \quad (21)$$

where $K^i K_i = R^{-2}$. Varying this free energy with respect to R and solving the equation of motion (4) leads to an equilibrium condition for the angular velocity [19]

$$\Omega = \Omega_{(0)} \left(1 + \frac{\sqrt{n+1}(3n+4)}{2n^2(n+2)} \xi(n) \varepsilon^2 \right), \quad (22)$$

where we have introduced the dimensionless order parameter $\varepsilon = r_0/R \ll 1$ and the leading order result

$$\Omega_{(0)} = \frac{1}{\sqrt{n+1}R}, \quad (23)$$

previously obtained in [10]. Using the thermodynamic expressions (14)-(16), which are now interpreted as the

mass, angular momentum and entropy of the black hole, we obtain

$$\frac{M}{\Omega_{(n+1)}(n+2)} = \frac{r_0^n}{8G} R \left(1 - \frac{(n+1)(3n+4)\xi\varepsilon^2}{2n^2(n+2)} \right), \quad (24)$$

$$\frac{J}{\Omega_{(n+1)}} = \frac{r_0^n}{8G} R^2 \sqrt{n+1} (1 + \mathcal{O}(\varepsilon^4)), \quad (25)$$

$$\frac{S n^{\frac{1}{2}}}{\Omega_{(n+1)}(n+1)^{\frac{1}{2}}} = \frac{\pi r_0^{n+1}}{2G} R \left(1 - \frac{(n+1)^2(3n+4)\xi\varepsilon^2}{2n^3(n+2)} \right) \quad (26)$$

where we have omitted the dependence of ξ on n . These thermodynamic expressions (24)-(26) are not invariant under field redefinitions (8). In order to present invariant results we introduce the reduced angular momentum, area, angular velocity and temperature as in [10],

$$j^{n+1} = c_j \frac{J^{n+1}}{GM^{n+2}}, \quad a_H^{n+1} = 4^{n+1} c_a \frac{S^{n+1}}{(GM)^{n+2}} \quad (27)$$

$$\omega_H = c_\omega \Omega (GM)^{\frac{1}{n+1}}, \quad t_H = c_t T (GM)^{\frac{1}{n+1}}, \quad (28)$$

with

$$c_j = \frac{(16\pi)^{n+1}}{2^{n+4} n^{\frac{n+1}{2}}} c_a = \frac{\Omega_{(n+1)}}{2^{n+5}} \frac{(n+2)^{n+2}}{(n+1)^{\frac{n+1}{2}}}, \quad (29)$$

$$c_\omega = \frac{\sqrt{n}}{4\pi} 8^{\frac{1}{n+1}} c_t = \sqrt{n+1} \left(\frac{n+2}{16} \Omega_{(n+1)} \right)^{-\frac{1}{n+1}}. \quad (30)$$

Using the freedom given by the field redefinition (8) we choose a gauge for which j does not receive any corrections to order ε^2 , this is done by performing the transformation $R \rightarrow R - \frac{(3n+4)}{2n^2} \xi(n) r_0 \varepsilon$. With this we obtain the improved phase structure of higher-dimensional black rings in dimensions $D \geq 7$,

$$a_H(j) = \frac{2^{\frac{n-2}{n(n+1)}}}{j^{\frac{1}{n}}} \left(1 + \frac{(n+1)(3n+4)}{2^{\frac{3n+4}{n}} n^3(n+2)} \frac{\xi(n)}{j^{\frac{2(n+1)}{n}}} \right), \quad (31)$$

$$\omega_H(j) = \frac{1}{2j} \left(1 + \frac{(n+1)(3n+4)}{2^{\frac{2(n+2)}{n}} n^2(n+2)} \frac{\xi(n)}{j^{\frac{2(n+1)}{n}}} \right), \quad (32)$$

$$t_H(j) = \frac{n j^{\frac{1}{n}}}{2^{\frac{n-2}{n(n+1)}}} \left(1 - \frac{3(n+1)(3n+4)}{2^{\frac{3n+4}{n}} n^3(n+2)} \frac{\xi(n)}{j^{\frac{2(n+1)}{n}}} \right). \quad (33)$$

The field redefinition invariant expressions (31)-(33) are one of the main results in this note and they describe the phase structure of higher-dimensional black rings beyond the infinitely thin approximation for which the corrections proportional to $\xi(n)$ in (31)-(33) are not present.

Below we compare the form of (31)-(33) with the infinitely thin approximation and with the numerical results recently obtained in [20] for $D = 7$ corresponding

to $n = 3$. We have also plotted the corresponding quantities for singly-spinning Myers-Perry (MP) black holes in $D = 7$. In Fig. 1 we exhibit the form of a_H as a function of j , extrapolating it to values of $j \sim \mathcal{O}(1)$. We see that the red curve given by Eq. (31) is in striking agreement with the blue curve obtained numerically even beyond the regime of validity $\varepsilon \ll 1$ of our analysis. Furthermore, we see that for a given value of the reduced angular momentum j the reduced area a_H increases compared to the infinitely thin limit. This indicates that including bending corrections in the free energy (2) leads to an increase in black hole entropy.

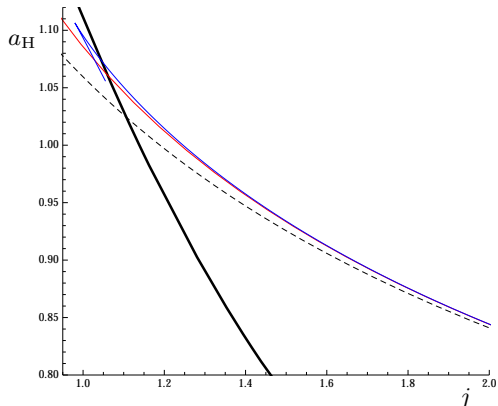


FIG. 1: a_H as a function of j . The solid black line corresponds to the MP black hole, while the dashed black line to the black ring in the infinitely thin limit [10]. The red curve is the improved phase structure given by (31) and the blue line is the numerically obtained curve for black rings in [20].

In Fig. 2 we plot the reduced angular velocity ω_H as a function of j . We see that the effects of bending require the black hole to have a higher angular velocity for a given value of j in order to compensate for the increase in the attractive force.

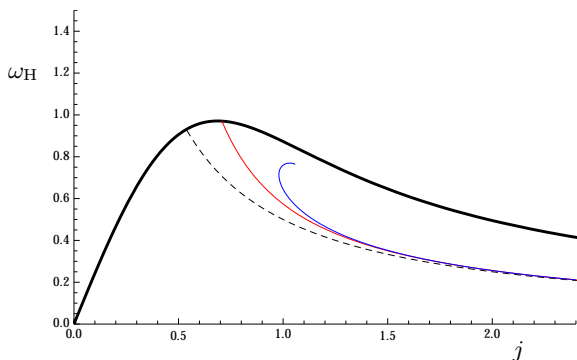


FIG. 2: ω_H as a function of j .

Finally, in Fig. 3 it is shown the reduced temperature t_H as a function of j . In this case the reduced temperature has decreased for a given value of j compared to

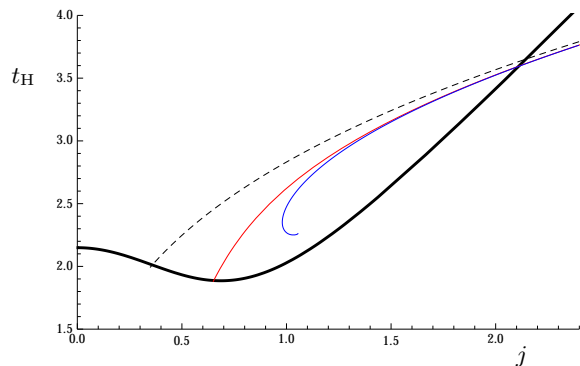


FIG. 3: t_H as a function of j .

the infinitely thin approximation. The behaviour of the expressions (31)-(33) exhibited in Figs. 1-3 is generic for any $D \geq 7$. It is therefore expected that the phase structure of black rings in higher dimensions is the same for any D as found numerically for $D = 6$ [21] and $D = 7$ [20, 22].

Discussion. In this note we have shown how the effective theory describing stationary black holes in the long-wavelength regime, also known as the blackfold approach [7, 8], can be used to obtain general features of higher-dimensional black rings. In such cases, the free energy of black rings (21) acquires an extra contribution which is the generalisation to higher codimension of the Helfrich-Canham bending energy (1). We see that in dimensions greater than six, the biophysical membrane effect is the leading contribution in a perturbative construction of higher-dimensional black rings, a statement which is verified for the first time by the numerical analysis of [20]. Furthermore, as in the case of biophysical membranes for which the effect of (1) is to increase their surface area, we see that black rings exhibit the same phenomena, increasing their area (and hence entropy) for a given value of angular momentum.

These bending effects are generic for wide classes of black holes, in particular, for all the new black holes found in [9], the effective theory (2) governs their dynamics and it will thus provide an improved description. In this context, it would be interesting to push the perturbative constructions of [10, 16] to next order so that the transport coefficients v_1, v_2, v_3 could be measured. We note that all our analysis has resided on the measurement of the Young modulus from a first order bent metric [3, 16] and we have shown that in the case of black rings, that data is enough to predict the phase structure to second order. The same techniques can be used for charged black holes and for black holes carrying transverse spin. These cases will be addressed in a future publication [18].

We believe that the analogy between gravity and biophysical membranes can have a fruitful development and that this opens up exciting new possibilities for applying

techniques used in biophysics to black holes and vice-versa.

Acknowledgements. We are grateful to Óscar J. C. Dias, Jorge E. Santos and Benson Way for sharing their unpublished numerical results for 7D black rings used to plot the blue curves in Figs.1-3. We also thank Joan Camps and Kentaro Tanabe for useful discussions. JA is supported by the Swiss National Science Foundation and the ‘Innovations- und Kooperationsprojekt C-13’ of the Schweizerische Universitätskonferenz SUK/CUS.

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